MATHEMATICS Proceedings A 85 (3), September 20, 1982

Sturm theory for certain classes of Sturm-Liouville equations and Turanians and Wronskians for the zeros of derivative of Bessel functions*

by Andrea Laforgia

Istituto di Calcoli Numerici dell'Università, Via Carlo Alberto 10, 10123 Torino, Italy

Communicated by Prof. C.J. Bouwkamp at the meeting of November 30, 1981

ABSTRACT

Let $y(x)$ be a non-trivial solution of the differential equation $y''+p(x)y=0$. In this paper we extend to the zeros of the derivative $y'(x)$ some known results on the zeros of $y(x)$.

In addition let $c'_{\nu k}$ be the kth zero of the derivative $C'_\nu(x)$ with respect to x of the general Bessel function $C_\nu(x)=AJ_\nu(x)+BY_\nu(x)$ where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of first and second kind, respectively. We show that $c'_{\nu,k+m}/c'_{\nu k}$ decreases to 1 as k increases, $m=1,2,\ldots$.

Finally we show that the Wronskian $W(c'_{\nu m}, \gamma'_{\nu k})<0$ if $\gamma'_{\nu k}<c'_{\nu m}$ (except in the case $c'_{\nu 1}>\nu>\gamma'_{\nu 1}$); here $c'_{\nu m}$ and $\gamma'_{\nu k}$ and zeros of the derivatives of two Bessel functions of the same order, not necessarily linearly independent.

1. INTRODUCTION AND BACKGROUND

Let x'_k be the kth zero of the derivative $y'(x)$ of a non-trivial solution of the differential equation

$$(1.1)\qquad y''+p(x)y=0.$$

In this paper we extend to x'_k some known results on the zeros of $y(x)$. Precisely we show that if $p(x)>0$ in (1.1) increases on the interval (a,b), then

$$x'_2-x'_1>x'_3-x'_2\ldots.$$

If $p(x)$ decreases on (a,b), the inequality sign in the result is reversed.

In addition we are concerned with the monotonicity in k of the quantity

* Research supported by Consiglio Nazionale delle Ricerche (Italy).

$c'_{\nu,k+m}/c'_{\nu k}$, where $c'_{\nu k}$ is the kth zero of the derivative of a general Bessel function; that is of $C'_\nu(x) = AJ'_\nu(x) + BY'_\nu(x)$ where the real numbers A, B are independent of x and ν, $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of first and second kind, respectively, and the prime indicates the derivative with respect to x. Precisely, we extend to $c'_{\nu k}$ some results obtained by Lorch [8] for the zeros $c_{\nu k}$ of the general Bessel function showing that

$$(1.2) \qquad \frac{c'_{\nu,k+m}}{c'_{\nu k}} \downarrow 1 \text{ as } k \text{ increases, } m = 1, 2, \ldots .$$

As in the analogous case in [8] we use (1.2) to prove the determinantal inequality

$$(1.3) \qquad T = \begin{vmatrix} c'_{\nu k} & c'_{\nu,k+1} \\ c'_{\nu,k+1} & c'_{\nu,k+2} \end{vmatrix} < 0.$$

Karlin and Szegö [4] named the determinants of this type Turánians.

Our interest is also in the Wronskian $W(\gamma'_{\nu k}, c'_{\nu m})$. We obtain a result analogous to that of Lorch [8] by proving that except in the case $c'_{\nu 1} > \nu > \gamma'_{\nu 1}$,

$$(1.4) \qquad W(\gamma'_{\nu k}, c'_{\nu m}) = \begin{vmatrix} \gamma'_{\nu k} & c'_{\nu m} \\ \gamma'^{(1)}_{\nu k} & c'^{(1)}_{\nu m} \end{vmatrix} < 0, \quad \nu \geq 0,$$

where the notation is arranged so that $\gamma'_{\nu k} < c'_{\nu m}$ and the superscript (1) indicates the derivative with respect to ν.

Our principal tool is the Sturm-Picone theorem and the main idea is to use what Lorch [7] called the simultaneous monotonicity in ν and x of the function $\phi_\nu(x)$ in a differential equation

$$y'' + \phi_\nu(x)y = 0.$$

In [1] and [5] it has been shown that this idea is useful for obtaining results on the spacing of the zeros of some classical orthogonal polynomials. But in the present case the monotonicity in ν and x of the function $\phi_\nu(x) > 0$ is used in the case of the differential equation

$$\left[\frac{z'}{\phi_\nu(x)}\right]' + z = 0.$$

In fact, given the self-adjoint linear differential equation

$$(1.5) \qquad [r(x)y']' + p(x)y = 0$$

where $r(x)$ and $p(x)$ are continuous and positive on an interval I, it is pointed out (compare for example [6]), that the self-adjoint differential equation

$$\left[\frac{z'}{p(x)}\right]' + \frac{z}{r(x)} = 0$$

is satisfied by $z(x) = r(x)y'(x)$, where $y(x)$ is any solution of (1.5).

For our purposes, the following form of the Sturm-Picone theorem will prove very useful; compare [14, pp. 2, 3] and [3, pp. 225–226].

LEMMA 1.1. (Sturm-Picone theorem). Given the self-adjoint linear second order differential equations

$$lu \equiv \frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = 0$$

$$Lv \equiv \frac{d}{dx}\left[A(x)\frac{dv}{dx}\right] + C(x)v = 0$$

on a bounded open interval $\alpha < x < \beta$ where $a(x)$, $A(x)$, $c(x)$, $C(x)$ are real-valued continuous functions and $a(x), A(x) > 0$ on $[\alpha, \beta]$; suppose $a(x) > A(x)$ and $c(x) < C(x)$ in the interval $\alpha < x < \beta$. If there exists a non-trivial real solution u of $lu = 0$ such that $u(\alpha) = u(\beta) = 0$, then every real solution of $Lv = 0$ has at least one zero in (α, β).

REMARK 1.1. Following the lines of the argument given in [14] it is possible to show that the conclusion of Lemma 1.1 remains valid if the hypothesis $c(x) < C(x)$ is replaced by $c(x) \leq C(x)$. However the theorem also holds if v is zero at one or both of α and β.

The integral representation [15, p. 510]

$$(1.6) \qquad \frac{dc'_{\nu k}}{d\nu} = \frac{2c'_{\nu k}}{c'^2_{\nu k} - \nu^2} \int_0^\infty (c'^2_{\nu k} \cosh 2t - \nu^2) K_0(2c'_{\nu k} \sinh t) c^{-2\nu t} dt$$

where $K_0(x)$ is the modified Bessel function of order zero, will be another important tool that we will use.

2. A GENERAL THEOREM

In this section we adapt to the zeros of $y'(x)$ Szegö's elegant formulation of a very simple consequence of the Sturm comparison theorem. It runs as follows [12, p. 186; 13, p. 20].

LEMMA 2.1. If $f(x)$ is monotonically increasing on the interval (a, b) and if a solution $y(x)$ of the differential equation $y'' + f(x)y = 0$ has consecutive zeros at x_1, x_2, x_3 on (a, b), then

$$x_3 - x_2 < x_2 - x_1.$$

If $f(x)$ decreases on (a, b) the inequality sign in the result is reversed.

REMARK 2.1. Incidentally we observe that E. Makai [10] has given a more general formulation of this Lemma involving the half-waves, while P. Hartman and A. Winter [2] found some results on the quarter-waves. Now we can enunciate the principal result of this section.

THEOREM 2.1. If $f(x) > 0$ increases on the interval (a, b) and if a solution $y(x)$ of the differential equation

$$(2.1) \qquad y'' + f(x)y = 0$$

is such that $y'(x)$ has consecutive zeros at x_1', x_2', x_3' on (a, b), then

(2.2) $$x_3'-x_2'<x_2'-x_1'.$$

If $f(x)>0$ decreases on (a, b) the inequality sign in the result is reversed.

PROOF. The differential equation

(2.3) $$\left[\frac{z'}{f(x)}\right]'+z=0$$

is satisfied by $z(x)=y'(x)$ where $y(x)$ is a solution of (2.1). Since $f(x)$ is an increasing function, we have $1/f(x)<1/f(x-d)$, where $d=x_2'-x_1'>0$. Now we compare (2.3) and the following equation,

$$\left[\frac{Z'}{f(x-d)}\right]'+Z(x)=0$$

satisfied by $Z=z(x-d)$. The function $Z(x)=z(x-d)$ has two consecutive zeros at x_2', $x_2'+d$, and the function $z(x)$ has two consecutive zeros at x_2', x_3'. Then an application of the Sturm-Picone theorem gives

$$x_3'<x_2'+d,$$

that is

$$x_3'-x_2'<x_2'-x_1'.$$

If $f(x)$ decreases, a similar argument shows that $x_2'-x_1'<x_3'-x_2'$, and this completes the proof.

Now we consider the differential equation

$$y''+\left[1-\frac{\nu^2-\frac{1}{4}}{x^2}\right]y=0$$

satisfied by the function $y=x^{\frac{1}{2}}C_\nu(x)$ and, in addition, the new differential equation

(2.4) $$\left[\frac{z'}{1-(\nu^2-\frac{1}{4})/x^2}\right]'+z=0$$

satisfied for $|\nu|\geq\frac{1}{2}$ by the function $z=y'(x)$.

Then we seek information about the differences of the zeros $t_{\nu k}$ and $x_{\nu k}$ of two solutions of (2.4). The proof of the following result runs along the same lines as that of theorem 2.1.

COROLLARY 2.1. Let $x_{\nu k}$ and $t_{\nu k}$ be the kth zeros of the functions $C_\nu(x)+$ $+2xC_\nu'(x)$ and $Z_\nu(x)+2xZ_\nu'(x)$ on the interval $[(\nu^2-\frac{1}{4})^{\frac{1}{2}}, \infty)$, where $C_\nu(x)$ and $Z_\nu(x)$ are two general Bessel functions, respectively. Then for $\varepsilon\geq 0$, $|\nu|>\frac{1}{2}$ and $x_{\nu+\varepsilon, k}<t_{\nu m}$ we have

$$t_{\nu, m+1}-t_{\nu m}<x_{\nu+\varepsilon, k+1}-x_{\nu+\varepsilon, k}.$$

REMARK 2.2. The result of corollary 2.1 has already been noted in [9, p. 362] as a consequence of a general, but more complicated theorem. Incidentally we observe that Muldoon [11] studied the monotonicity of the difference of the zeros of the more general function $\mu J_\nu(x)+xJ_\nu'(x)$ which reduces to our function $J_\nu(x)+2xJ_\nu'(x)$ when $\mu=\frac{1}{2}$.

3. MONOTONIC RESULTS AND TURÁNIANS

The differential equation

$$y''+(e^{2x}-\nu^2)y=0$$

is satisfied by the function $y_\nu=C_\nu(e^x)$ [15, p. 99], hence for $x>\log|\nu|$ (see the introduction) the function $z_\nu=y_\nu'(x)$ is a solution of the differential equation

$$\left[\frac{z'}{e^{2x}-\nu^2}\right]'+z=0. \tag{3.1}$$

Let $C_\nu(x)$ and $Z_\nu(x)$ be general Bessel functions not necessarily linearly independent, with $y_{\nu k}'$ and $z_{\nu k}'$ denoting the zeros of the derivatives $C_\nu'(x)$ and $Z_\nu'(x)$, respectively, where for a fixed pair of positive integers k and m

$$z_{\nu+\varepsilon,k}'<y_{\nu m}' \tag{3.2}$$

for $\varepsilon\geq 0$.

Besides (3.1) we consider the differential equation

$$\left[\frac{w'}{e^{2(x-h)}-(\nu+\varepsilon)^2}\right]'+w=0 \tag{3.3}$$

for $x>\log|\nu+\varepsilon|+h$, satisfied by the function $w_\nu(x)=z_{\nu+\varepsilon}(x-h)$ where $h=\log y_{\nu m}'-\log z_{\nu+\varepsilon,k}'$ is positive in accordance with (3.2). In the notation of the Lemma 1.1 we have

$$A(x)=\frac{1}{e^{2x}-\nu^2},\ a(x)=\frac{1}{e^{2(x-h)}-(\nu+\varepsilon)^2},\quad c(x)=C(x)=1.$$

The functions $z_\nu(x)$ and $z_{\nu+\varepsilon}(e^{x-h})$ both vanish at $x=\log y_{\nu m}'$ and it is easy to see that $a(x)>A(x)$.

Then the next zero of $C_\nu'(e^x)$ precedes, by Lemma 1.1, the next zero of $C_{\nu+\varepsilon}'(e^{x-h})$. Hence we have $\log y_{\nu,m+N}'<\log z_{\nu+\varepsilon,k+N}'+h$, $(N=1,2,\ldots)$ so that

$$\frac{y_{\nu,m+N}'}{y_{\nu m}'}<\frac{z_{\nu+\varepsilon,k+N}'}{z_{\nu+\varepsilon,k}'},\quad \nu\geq 0.$$

This establishes the following result.

THEOREM 3.1. Let $y_{\nu k}'$ and $z_{\nu k}'$ be the kth zeros of the derivatives of the general Bessel functions $C_\nu(x)$ and $Z_\nu(x)$, respectively, and let for a fixed pair of positive integers k and m

$$z_{\nu+\varepsilon,k}'<y_{\nu m}',\quad \varepsilon\geq 0.$$

Then

$$(3.4)\qquad \frac{\gamma'_{\nu,m+N}}{\gamma'_{\nu m}}<\frac{z'_{\nu+\varepsilon,k+N}}{z'_{\nu+\varepsilon,k}},\quad N=1,2,\dots,\ \nu\geq 0.$$

REMARK 3.1. As a particular case of (3.4) we have for $\varepsilon=0$, $z'_{\nu k}=\gamma'_{\nu k}=c'_{\nu k}$, $m=k+1$, $N=1$

$$\frac{c'_{\nu,k+2}}{c'_{\nu,k+1}}<\frac{c'_{\nu,k+1}}{c'_{\nu k}},\quad \nu\geq 0,\quad k=1,2,\dots,$$

that is

$$\frac{c'_{\nu,k+m}}{c'_{\nu k}}\downarrow \text{ with } k\to\infty,$$

from which the result (1.3) follows.

The asymptotic formula [15, p. 507]

$$c'_{\nu m}=(n+\nu/2+\tfrac{1}{4})\pi-\alpha+O(n^{-1}),\quad \nu\geq 0$$

where α is independent of n and ν, can be used to prove the more precise result

$$\frac{c'_{\nu,k+m}}{c'_{\nu k}}\downarrow 1 \text{ with } k\to\infty.$$

4. THE WRONSKIAN

In this section it will be shown that for $\gamma'_{\nu k}<c'_{\nu m}$

$$(4.1)\qquad W(\gamma'_{\nu k},c'_{\nu m})<0,\quad \nu\geq 0.$$

We indicate the derivative with respect to ν of $\gamma'_{\nu k}$ and $c'_{\nu m}$ by $\gamma'^{(1)}_{\nu k}$ and $c'^{(1)}_{\nu m}$, respectively. With this notation the Wronskian is

$$W(\gamma'_{\nu k},c'_{\nu m})=\begin{vmatrix}\gamma'_{\nu k} & c'_{\nu m}\\ \gamma'^{(1)}_{\nu k} & c'^{(1)}_{\nu m}\end{vmatrix}$$

The integral formula (1.6) gives

$$W(\gamma'_{\nu k},c'_{\nu m})=\gamma'_{\nu k}c'^{(1)}_{\nu m}-c'_{\nu m}\gamma'^{(1)}_{\nu k}=$$

$$=2\gamma'_{\nu k}c'_{\nu m}\int_0^\infty\left\{\frac{c'^2_{\nu m}\cosh 2t-\nu^2}{c'^2_{\nu m}-\nu^2}K_0(2c'_{\nu m}\sinh t)-\right.$$

$$\left.-\frac{\gamma'^2_{\nu k}\cosh 2t-\nu^2}{\gamma'^2_{\nu k}-\nu^2}K_0(2\gamma'_{\nu k}\sinh t)\right\}e^{-2\nu t}dt.$$

Since the function $K_0(x)$ decreases with x, to prove (4.1) it will be sufficient to show that

$$\frac{c'^2_{\nu m}\cosh 2t-\nu^2}{c'^2_{\nu m}-\nu^2}<\frac{\gamma'^2_{\nu k}\cosh 2t-\nu^2}{\gamma'^2_{\nu k}-\nu^2}.$$

But the last inequality can be written (except in the case $c'_{\nu 1} > \nu > \gamma'_{\nu 1}$) in the form

$$(4.2) \qquad (c'^2_{\nu m} - \gamma'^2_{\nu k}) \cosh 2t > c'^2_{\nu m} - \gamma'^2_{\nu k}.$$

The validity of (4.2) establishes (4.1).

ACKNOWLEDGMENT

We are grateful to Professor Martin E. Muldoon for his interest and for useful comments.

REFERENCES

1. Ahmed, S., A. Laforgia and M.E. Muldoon – On the spacing of the zeros of some classical orthogonal polynomials, J. London Math. Soc. (2), **25**, 246–252 (1982).
2. Hartman, P. and A. Winter – On non conservative linear oscillators low frequency, Amer. J. Math., **70**, 529–539 (1945).
3. Ince, E.L. – Ordinary differential equations, Dover reprint, 1944.
4. Karlin, S. and G. Szegö – On certain determinants whose elements are orthogonal polynomials, J. d'Analyse Math., **8**, 1–157 (1960).
5. Laforgia, A. – Monotonic properties for the zeros of ultraspherical polynomials, Proc. Am. Math. Soc. (4), **83**, 757–758 (1981).
6. Leighton, W. – Some Sturm theory, J. d'Analyse Math., **36**, 191–197 (1979).
7. Lorch, L. – Elementary comparison techniques for certain classes of Sturm-Liouville equations, Proc. Uppsala 1977 Inter. Conf. Diff. Equations, Symposia Univ. Upsaliensis Annum Quingentesimum celebrantis 7, Acta Univ. Upsaliensis, Uppsala 125–133 (1977).
8. Lorch, L. – Turánians and Wronskians for the zeros of Bessel functions, SIAM J. Math. An. **11**, 223–227 (1980).
9. Lorch, L., M.E. Muldoon and P. Szegö – Higher monotonicity properties of certain Sturm-Liouville functions, IV Canad. J. Math. **24**, 349–365 (1972).
10. Makai, E. – On a monotonic property of certain Sturm-Liouville functions, Acta Math. Acad. Sci. Hungar. **3**, 165–172 (1952).
11. Muldoon, M.E. – On the zeros of a function related to Bessel functions, Arch. Math. (Brno), (to appear).
12. Sansone, G. – Equazioni differenziali nel campo reale, Zanichelli Editore Bologna 1941.
13. Szegö, G. – Orthogonal Polynomials, 4th ed., Amer. Math. Soc. Colloquium Pubblications **23**, Amer. Math. Soc. RI, 1975.
14. Swanson, C.A. – Comparison and oscillation theory of linear differential equations, Academic Press, 1968.
15. Watson, G.N. – A treatise on the theory of Bessel functions, 2nd ed., Cambridge University Press, Cambridge, 1958.